A Theory for Stress Analysis of Composite Laminates

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Failures in laminated resin matrix composite materials often begin with matrix microcracking and delamination. These modes of damage are three-dimensional in nature and are controlled by interlaminar stresses. One important key to understanding and ultimately predicting the failures in composite materials is an analytical approach that provides reliable stress estimates in critical regions. Conventional laminate theories are inadequate for this purpose as they are based on global displacement assumptions. Moreover, the interlaminar stresses are often neglected in the initial formulations. Therefore, solutions based upon these theories cannot yield realistic stress distributions. Recent theoretical research shows that there are certain nonclassical influences that affect bending-related behavior. They include section warping and its concomitant nonclassical surface-parallel stress contributions and transverse normal strain. The stress prediction capability of a bending theory improves significantly if these nonclassical influences are incorporated. A comprehensive bending theory is developed for arbitrary composite laminates. Its effectiveness is demonstrated in examples for a cross-ply laminate and a quasi-isotropic laminate.

Introduction

THE directional nature of composite laminates poses unique challenges for the analyst. The extensional modulus along the direction of fibers is usually very large relative to the extensional moduli in the lateral directions and the shear moduli. This is a marked departure from conventional isotropic materials. Consequently, the relative importance of physical effects is influenced by the directional nature of properties and their relative magnitude. Transverse shear deformations, for example, are much more pronounced for composite structures.

Laminated plate theories based upon the Kirchhoff hypothesis have been developed by Reissner and Stavsky[1] and Dong, Pister, and Taylor.[2] Subsequently, Whitney and Leissa[3] introduced the influence of in-plane mass and rotatory inertia for dynamics and large deflections. These classical theories are adequate for the prediction of overall response characteristics of thin laminates in most applications.

Stavsky[4] introduced the influence of transverse shear deformation into a laminated plate theory. Laminates made from isotropic layers of material were considered. This work was later extended to generally anisotropic laminates by Yang, Norris, and Stavsky[5] and Whitney and Pagano.[6]

The shear deformation theories of Refs. 4-6 are similar to Mindlin's work[7] on isotropic plates. Shear correction factors are introduced and selected to achieve good agreement in some specified benchmark problem.[6-10] As illustrated in Refs. 6 and 8, these factors depend upon the constituent ply properties, ply layup, fiber orientation, and the particular application. This type of approach, therefore, is semi-empirical.

Further developments in laminate analysis have been higher-order theories.[11-14] "Higher-order" denotes a variation of displacement that is higher than linear through the laminate thickness. A discussion of these theories appears in Ref. 15. In general, their application is far more difficult than that of both the classical and transverse shear deformation theories.

Displacement formulations of the above type begin with kinematically admissible displacements, but the stress equilibrium equations are violated. The stresses will be discontinuous at laminar interfaces. In general, an assumed displacement approach yields relatively poor stress estimates. Alternative means may have to be sought to improve stresses. These include use of equilibrium equations for obtaining interlaminar stresses, as in Ref. 16. Whitney[10] uses the shear stresses in a second evaluation of surface parallel stresses.

References 12, 17, and 18 provide theories for laminates having highly dissimilar plies. They rely on piecewise linear displacements. As a result, the shear stresses become discontinuous at the laminar interfaces. In spite of the accurate results obtained for phase velocities and in-plane stresses, the theories are not generally suitable for laminate stress analysis.

Recent theoretical research[19] has contributed a new appreciation for nonclassical factors in addition to transverse shear deformation in bending-related behavior. A nonclassical contribution to axial stress and transverse normal strain significantly affect the response in planar-bending situations for certain combinations of geometry and stiffness in homogeneous structures. Incorporating these influences, theories were developed for homogeneous plates,[20] stiffened plates,[21] and the planar bending of laminates.[22] Stresses and displacements are improved beyond linear distributions, yet the overall equations retain the character of a standard shear deformation theory. This is a distinguishing feature, for it allows the integration of nonclassical influences on a simple basis.

The present work has been undertaken as a result of the above observations. A new, comprehensive theory for the bending of composite laminates is presented and applied. Be-

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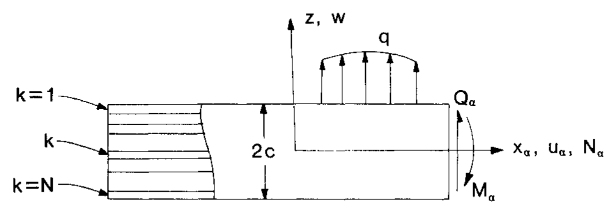

Fig. 1 Notation, coordinates and numbering of plies.

ing simple, consistent, and complete, the theory rationally accounts for the usually important nonclassical influences. The fundamental ideas of Refs 19-22 are the cornerstone of the development, and the solutions of a benchmark problem illustrate the significance of the nonclassical influences on behavior. Also, comparative studies have evaluated the results from the present and earlier theories.

Overview

Classical Kirchhoff plate theory is based upon kinematic assumptions that are equivalent to ignoring, and hence setting to zero, the transverse normal strain and the two transverse shear strain components. The central assumption that replaces the Kirchhoff hypothesis in the present development is that the statically equivalent stresses obtained from classical theory can be used to estimate transverse normal and shear strains. These strain estimates permit the form of the displacement field to be determined in each lamina. Continuity of interlaminar tractions and displacement at interlaminar boundaries is maintained. The displacement field is subsequently used to improve the expression for surface-parallel stresses. The development is a logical extension of that of Ref. 22.

A brief summary of classical theory follows. It serves to introduce the notation and provide the basis for the new theory.

Summary of Classical Laminated Plate Theory

Consider a laminate made of N perfectly bonded anisotropic layers, each having a plane of material symmetry parallel to the plane of the laminate. Let k denote a particular ply singled out for study. The notation and sign convention and order of the numbering of the plies are shown in Fig. 1. The reference surface $z=0$ may be located arbitrarily. To facilitate the presentation, the following subscript convention is used: Latin indices will assume the values 1, 2, and 6; Greek indices will assume the values 1 and 2; and repeated indices in the same expression imply a sum of terms over the appropriate range of the index.

The stress components are identified in the conventional manner: σ_{11}, σ_{22}, σ_{zz}, σ_{2z}, σ_{1z}, and σ_{12}. An alternative notation is to write the stresses in the same order with a single subscript: σ_1, σ_2, σ_3, σ_4, σ_5, and σ_6.

The differential equations of motion that are valid within each ply are

$$\sigma^k_{\alpha\beta,\beta}+\sigma^k_{\alpha z,z}=\rho^k\ddot{u}^k_\alpha \tag{1}$$

$$\sigma^k_{\beta z,\beta}+\sigma^k_{zz,z}=\rho^k\ddot{w}^k \tag{2}$$

where ρ^k is the mass density of the kth ply. The above equations are supplemented by interlaminar traction and displacement continuity conditions. The traction boundary conditions specified at the extreme upper surfaces of the laminates are

$$\text{At } z=c\text{:}\quad \sigma_{zz}=q,\quad \sigma_{1z}=0,\quad \sigma_{2z}=0$$

$$\text{At } z=-c\text{:}\quad \sigma_{zz}=0,\quad \sigma_{1z}=0,\quad \sigma_{2z}=0$$

Overall equations of motion describing the gross behavior of the laminate may be derived by appropriate integration of the ply equations and adding the results.

$$N_{\beta\alpha,\alpha}=m\ddot{\bar{U}}_\beta+s\ddot{\bar{\phi}}_\beta \tag{3}$$

$$M_{\beta\alpha,\alpha}-Q_\beta=s\ddot{\bar{U}}_\beta+I\ddot{\bar{\phi}}_\beta \tag{4}$$

$$Q_{\beta,\beta}+q=m\ddot{\bar{W}} \tag{5}$$

The usual notation applies for the generalized force variables,

$$(N_{\alpha\beta},M_{\alpha\beta},Q_\alpha)=\Sigma\int_k(\sigma^k_{\alpha\beta},z\sigma^k_{\alpha\beta},\sigma^k_{\alpha z})\mathrm{d}z \tag{6}$$

The symbol Σ denotes a summation over all of the plies. The generic ply symbol k used with an integral denotes integration over the thickness of the kth ply. In addition,

$$(m,s,I)=\Sigma\int_k(1,z,z^2)\rho^k\mathrm{d}z \tag{7}$$

where m is the mass per unit area of the laminate, s the static unbalance per unit area about an axis in the reference plane, and I the rotatory inertia per unit area about an axis in the reference plane. For laminates with physical symmetry about the reference surface, $s=0$, the bending and stretching motions are uncoupled. However, this requires that the reference surface, $z=0$, be located at the laminate middle surface.

The kinematic variables are related to the actural displacements by the following relations:

$$m\bar{U}_\beta+s\bar{\phi}_\beta=\Sigma\int_k\rho^k u_\beta\mathrm{d}z \tag{8}$$

$$s\bar{U}_\beta+I\bar{\phi}_\beta=\Sigma\int_k\rho^k u_\beta z\mathrm{d}z \tag{9}$$

$$m\bar{W}=\Sigma\int_k\rho^k w\mathrm{d}z \tag{10}$$

Equations (3-5) are a central focus in classical and shear deformation theories. They are also adopted as equations of motion in the present theory.

Classical Displacement

In classical theory, the displacement field is constrained by Kirchhoff's hypothesis and is linear through the thickness of the laminate,

$$u_\beta=U_\beta-zW_{,\beta} \tag{11}$$

$$w=W \tag{12}$$

where U_1, U_2, and W are displacement components associated with the reference surface. For this distribution, the kinematic variables become

$$\bar{U}_\beta=U_\beta \tag{13}$$

$$\bar{\phi}_\beta=-W_{,\beta} \tag{14}$$

$$\bar{W}=W \tag{15}$$

The classical displacement assumption intrinsically ensures interlaminar kinematic compatibility throughout the laminate. A consistent, relatively simple theory is the result. However, its inability to predict effects is the motivation for this development.

Classical Stresses

Classical laminated plate theory provides the following expressions for the surface-parallel stresses:

$$\sigma^k_i=n^k_{ij}N_j+m^k_{ij}M_j \tag{16}$$

where

$$n^k_{ij}=\bar{A}^k_{ij}+z\bar{B}^k_{ij} \tag{17}$$

$$m_{ij}^k = \bar{B}_{ij}^k + z\bar{D}_{ij}^k \tag{18}$$

with the following directions adopted for the ply dependent matrices:

$$(\bar{A}_{ij}^k, \bar{B}_{ij}^k, \bar{D}_{ij}^k) = \bar{C}_{i\ell}^k (A_{\ell j}^*, B_{\ell j}^*, D_{\ell j}^*) \tag{19}$$

where $[C^k]$ is the plane stress stiffness matrix for the kth ply and $[A^*]$, $[B^*]$, and $[D^*]$ denote the extensional, coupling, and bending flexibility matrices, respectively. The quantities n_{ij}^k and m_{ij}^k physically correspond to the axial stresses σ_{ij}^k in the kth ply due to a unit in-plane force resultant N_j and moment resultant M_j, respectively. These are functions of the thickness coordinate only.

Estimates for the transverse shear stresses and the transverse normal stress are obtained by elementary means. Equations (16), (11), and (12) provide the distribution of the surface-parallel stresses and displacements through the thickness. With the aid of these distributions, the equations of motion (1) and (2) are integrated to arrive at the following expressions for the transverse stresses:

$$\sigma_{1z}^k = -\tilde{n}_{1j}^k N_{j,1} - \tilde{n}_{6j}^k N_{j,2} - \tilde{m}_{1j}^k M_{j,1} - \tilde{m}_{6j}^k M_{j,2} - \tilde{t}^k Q_1 \tag{20}$$

$$\sigma_{2z}^k = -\tilde{n}_{6j}^k N_{j,1} - \tilde{n}_{2j}^k N_{j,2} - \tilde{m}_{6j}^k M_{j,1} - \tilde{m}_{2j}^k M_{j,2} - \tilde{t}^k Q_2 \tag{21}$$

$$\begin{aligned} \sigma_{zz}^k = {} & \tilde{n}_{1j}^k N_{j,11} + 2\tilde{n}_{6j}^k N_{j,12} + \tilde{n}_{2j}^k N_{j,22} + \tilde{m}_{1j}^k M_{j,11} \\ & + 2\tilde{m}_{6j}^k M_{j,12} + \tilde{m}_{2j}^k M_{j,22} + \tilde{t}^k Q_{\beta,\beta} + \tilde{t}_1^k q \end{aligned} \tag{22}$$

where $\tilde{n}_{ij}^k, \tilde{m}_{ij}^k, \ldots$ are functions of the thickness coordinate and material properties, respectively. They define the thickness variation of the transverse stresses in the kth ply. In obtaining these expressions, the overall equations of motion (3-5) are used to eliminate the acceleration terms. The traction continuity is also satisfied at the laminar interfaces. This procedure is described in Refs. 22 and 23. The final expressions are given in the Appendix.

Foundations of a New Theory

The fundamental assumption that permits the development of the present theory is that the transverse normal and shear strains can be estimated from the statically equivalent stresses given by Eqs. (16) and (20-22). An additional assumption is that the influence of the transverse normal stress can be neglected in estimating the transverse normal strain. These two assumptions replace the Kirchhoff hypothesis in the present development. Their significance has been discussed in Refs. (18-23).

Kinematics

Hooke's law for the transverse normal strain in each ply is

$$w_{,z}^k = S_{3i}^k \sigma_i^k + \underline{S_{33}^k \sigma_{zz}^k} \tag{23}$$

The S parameters are flexibilities that may vary from ply to ply. The underlined term is neglected in accordance with the assumption just made. With the aid of the stress distributions of Eq. (16), this strain is integrated in each ply and the arbitrary integration parameters are related to the reference surface deflection w. The final result may be written in the following form:

$$w^k(x_1, x_2, z, t) = W(x_1, x_2, t) + \eta_i^k(z) N_i + \mu_i^k(z) M_i \tag{24}$$

The functions of the thickness coordinate η_i^k and μ_i^k are defined in the Appendix.

The form of the in-plane displacements is determined in an analogous manner beginning with the stress-strain displacement relations:

$$u_{1,z}^k = -w_{,1}^k + S_{54}^k \sigma_{2z}^k + S_{55}^k \sigma_{1z}^k \tag{25}$$

$$u_{2,z}^k = -w_{,2}^k + S_{44}^k \sigma_{2z}^k + S_{45}^k \sigma_{1z}^k \tag{26}$$

Equations (20), (21), and (24) provide

$$\begin{aligned} u_{1,z}^k = {} & -W_{,1} - (\eta_i + S_{55}\tilde{n}_{1i} + S_{54}\tilde{n}_{6i})^k N_{i,1} - S_{55}^k \tilde{t}^k Q_1 \\ & - (S_{55}\tilde{n}_{6i} + S_{54}\tilde{n}_{2i})^k N_{i,2} - (\mu_i + S_{55}\tilde{m}_{1i} + S_{54}\tilde{m}_{6i})^k M_{i,1} \\ & - S_{54}^k \tilde{t}^k Q_2 - (S_{55}\tilde{m}_{6i} + S_{54}\tilde{m}_{2i})^k M_{i,2} \end{aligned} \tag{27}$$

$$\begin{aligned} u_{2,z}^k = {} & -W_{,2} - (S_{54}\tilde{n}_{1i} + S_{44}\tilde{n}_{6i})^k N_{i,1} \\ & - (\eta_i + S_{54}\tilde{n}_{6i} + S_{44}\tilde{n}_{2i})^k N_{i,2} - (S_{54}\tilde{m}_{1i} + S_{44}\tilde{m}_{6i})^k M_{i,1} \\ & - (\mu_i + S_{54}\tilde{m}_{6i} + S_{44}\tilde{m}_{2i})^k M_{i,2} - S_{54}^k \tilde{t}^k Q_1 - S_{44}^k \tilde{t}^k Q_2 \end{aligned} \tag{28}$$

It should be noted that the superscript k applies to all quantities within the parentheses. The derivatives are integrated by enforcing the continuity conditions as described in Ref. 23. The final results are

$$\begin{aligned} u_\alpha^k = {} & U_\alpha - zW_{,\alpha} - u_{1_{\alpha i}}^k N_{i,1} - u_{2_{\alpha i}}^k N_{i,2} - u_{5_{\alpha\beta}}^k Q_\beta \\ & - u_{3_{\alpha i}}^k M_{i,1} - u_{4_{\alpha i}}^k M_{i,2} \end{aligned} \tag{29}$$

U_1 and U_2 are the displacements of the reference surface. The functions of the thickness coordinate are determined in a manner similar to η_{ij}^k and μ_{ij}^k.

Refined Surface-Parallel Stresses

The new displacements are used in conjunction with the stress-strain relations,

$$\sigma_i^k = \bar{c}_{ij}^k \epsilon_j^k + \bar{c}_i^k \sigma_{zz}^k \tag{30}$$

to estimate the surface-parallel stresses. The constants $\bar{c}_1^k$, $\bar{c}_2^k$, and $\bar{c}_6^k$ introduce the influence of the transverse stress in the computation of the refined surface-parallel stresses. The expression for these constants in terms of ply material properties can be obtained when the generalized Hookean relations for ϵ_1, ϵ_2, and γ_6 are inverted. The classical plane stress assumption is the equivalent to neglecting $\bar{c}_1$, $\bar{c}_2$, and $\bar{c}_6$. Together with the transverse normal stress [Eq. (22)], the displacements provide the following:

$$\begin{aligned} \sigma_i^k = {} & \bar{c}_{ij}^k (\epsilon_j^0 + z\chi_j^0) + S_{1_{ij}}^k N_{j,11} + S_{2_{ij}}^k N_{j,12} + S_{3_{ij}}^k N_{j,22} + S_{4_{ij}}^k M_{j,11} \\ & + S_{5_{ij}}^k M_{j,12} + S_{6_{ij}}^k M_{j,22} + S_{7_{i\beta}}^k Q_{\beta,1} + S_{8_{i\beta}}^k Q_{\beta,2} + S_{9_i}^k q \end{aligned} \tag{31}$$

where

$$S_{1_{ij}}^k = (\bar{c}_i \tilde{n}_{1j} - \bar{c}_{i1} u_{1_{1j}} - \bar{c}_{i6} u_{1_{2j}})^k$$

$$S_{2_{ij}}^k = (2\bar{c}_i \tilde{n}_{6j} - \bar{c}_{i1} u_{2_{1j}} - \bar{c}_{i2} u_{1_{2j}} - \bar{c}_{i6} u_{1_{1j}} - \bar{c}_{i6} u_{2_{2j}})^k$$

$$S_{3_{ij}}^k = (\bar{c}_i \tilde{n}_{2j} - \bar{c}_{i6} u_{2_{ij}} - \bar{c}_{i2} u_{2_{2j}})^k$$

$$S_{4_{ij}}^k = (\bar{c}_i \tilde{m}_{1j} - \bar{c}_{i1} u_{3_{1j}} - \bar{c}_{i6} u_{3_{2j}})^k$$

$$S_{5_{ij}}^k = (2\bar{c}_i \tilde{m}_{6j} - \bar{c}_{i1} u_{4_{ij}} - \bar{c}_{i6} u_{3_{ij}} - \bar{c}_{i2} u_{3_{2j}} - \bar{c}_{i6} u_{4_{2j}})^k$$

$$S_{6_{ij}}^k = (\bar{c}_i \tilde{m}_{2j} - \bar{c}_{i2} u_{4_{2j}} - \bar{c}_{i6} u_{4_{1j}})^k$$

$$S_{7_{i1}}^k = (\bar{c}_i \tilde{t} - \bar{c}_{i1} u_{5_{11}} - \bar{c}_{i6} u_{5_{21}})^k$$

$$S_{7_{i2}}^k = (-\bar{c}_{i1} u_{5_{12}} - \bar{c}_{i6} u_{5_{22}})^k$$

$$S_{8_{i1}}^k = (-\bar{c}_{i2} u_{5_{21}} - \bar{c}_{i6} u_{5_{11}})^k$$

$$S_{8_{i2}}^k = (\bar{c}_i \tilde{t} - \bar{c}_{i2} u_{5_{22}} - \bar{c}_{i6} u_{5_{12}})^k$$

$$S_{9_i}^k = \bar{c}_i^k \tilde{t}_1^k \tag{32}$$

The functions (S_1^k-S_9^k) describe the thickness variation of the stresses. The deflections of the reference surface (U_1, U_2, and w) must be related to the stress and moment resultants. This is accomplished by enforcing the definitions in Eq. (6). The resultants are

$$N_i = A_{ij}\epsilon_j^0 + B_{ij}\chi_j^0 + K_{1_{ij}}^0 N_{j,11} + K_{2_{ij}}^0 N_j, 12 + K_{3_{ij}}^0 N_{j,22}$$
$$+ K_{4_{ij}}^0 M_{j,11} + K_{5_{ij}}^0 M_{j,12} + K_{6_{ij}}^0 M_{j,22} + K_{7_{i\beta}}^0 Q_{\beta,1}$$
$$+ K_{8_{i\beta}}^0 Q_{\beta,2} + K_{9_i}^0 q \tag{33}$$

$$M_i = B_{ij}\epsilon_j^0 + D_{ij}\chi_j^0 + K_{1_{ij}}^1 N_{j,11} + K_{2_{ij}}^1 N_{j,21} + K_{3_{ij}}^1 N_{j,22}$$
$$+ K_{4_{ij}}^1 M_{j,11} + K_{5_{ij}}^1 M_{j,12} + K_{6_{ij}}^1 M_{j,22} + K_{7_{i\beta}}^1 Q_{\beta,1}$$
$$+ K_{8_{i\beta}}^1 Q_{\beta,2} + K_{9_i}^1 q \tag{34}$$

where

$$(K_{\ell_{ij}}^0, K_{\ell_{ij}}^1) = \Sigma\int_k (1,z) S_{\ell_{ij}}^k \mathrm{d}z; \quad i = 1\text{-}6$$

$$(K_{\ell_{i\beta}}^0, K_{\ell_{i\beta}}^1) = \Sigma\int_k (1,z) S_{\ell_{i\beta}}^k \mathrm{d}z; \quad \ell = 7,8$$

$$(K_{9_i}^0, K_{9_i}^1) = \Sigma\int_k (1,z) S_{9_i}^k \mathrm{d}z \tag{35}$$

$[A]$, $[B]$, and $[D]$ are the plane stress extensional, coupling, and bending stiffness matrices, respectively, and $\{\epsilon^0\}$ and $\{\kappa^0\}$ the extensional strain and curvature vectors of the reference surface. The surface-parallel stresses may be expressed in terms of generalized force variables with the aid of Eqs. (33), (34), (17), and (18). The expressions are

$$\sigma_i^k = \underline{n_{ij}^k N_j + m_{ij}^k M_j} + (S_{1_\ell} - n_{ij}K_{1_{j\ell}}^0 - m_{ij}K_{1_{j\ell}}^1)^k N_{\ell,11}$$
$$+ (S_{2_{i\ell}} - n_{ij}K_{2_{j\ell}}^0 - m_{ij}K_{2_{j\ell}}^1)^k N_{\ell,12}$$
$$+ (S_{3_{i\ell}} - n_{ij}K_{3_{j\ell}}^0 - m_{ij}K_{3_{j\ell}}^1)^k N_{\ell,22}$$
$$+ (S_{4_{i\ell}} - n_{ij}K_{4_{j\ell}}^0 - m_{ij}K_{4_{j\ell}}^1)^k M_{\ell,11}$$
$$+ (S_{5_{i\ell}} - n_{ij}K_{5_{j\ell}}^0 - m_{ij}K_{5_{j\ell}}^1)^k M_{\ell,12}$$
$$+ (S_{6_{i\ell}} - n_{ij}K_{6_{j\ell}}^0 - m_{ij}K_{6_{j\ell}}^1)^2 M_{\ell,22}$$
$$+ (S_{7_{i\alpha}} - n_{ij}K_{7_{j\alpha}}^0 - m_{ij}K_{7_{j\alpha}}^1)^k Q_{\alpha,1}$$
$$+ (S_{8_{i\alpha}} - n_{ij}K_{8_{j\alpha}}^0 - m_{ij}K_{8_{j\alpha}}^1)^k Q_{\alpha,2}$$
$$+ (S_{9_i} - n_{ij}K_{9_j}^0 - m_{ij}K_{9_j}^1)^k q \tag{36}$$

The superscript k applies to all quantities in the parentheses, except the K stiffness parameters. The underlined terms correspond to the classical result. The remaining terms are associated with the local distributions that are nonlinear in each ply. They correspond to deformations that are nonlinear through the thickness of the laminate and that cannot be predicted by classical theory. Their determination constitutes the desired refinement.

It has been shown in Ref. 23 that the present stresses become exact in planar bending for the case of uniformly distributed load. This may be considered as a point in support of the procedure adopted for the development of the stresses.

Overall Equations of Motion

Equations (3-5) still govern the overall motion of the laminate. However, the kinematic variables appearing in these equations should be computed anew using the displacement distributions of Eqs. (24) and (29). The following results are obtained as a result of solving the equations for $\bar{U}_\alpha$, $\bar{\phi}_\alpha$, $\bar{W}$:

$$\bar{U}_\alpha = U_\alpha - (m^* J_{1_{\alpha i}}^0 + s^* J_{1_{\alpha i}}^1) N_{i,1}$$
$$- (m^* J_{2_{\alpha i}}^0 + s^* J_{2_{\alpha i}}^1) N_{i,2} - (m^* J_{3_{\alpha i}}^0 + s^* J_{3_{\alpha i}}^1) M_{i,1}$$
$$- (m^* J_{4_{\alpha i}}^0 + s^* J_{4_{\alpha i}}^1) M_{i,2} - (m^* J_{5_{\alpha\beta}}^0 + s^* J_{5_{\alpha\beta}}^1) Q_\beta \tag{37}$$

$$\bar{\phi}_\alpha = -w_{,\alpha} - (s^* J_{1_{\alpha i}}^0 + I^* J_{1_{\alpha i}}^1) N_{i,1}$$
$$- (s^* J_{2_{\alpha i}}^0 + I^* J_{2_{\alpha i}}^1) N_{i,2} - (s^* J_{3_{\alpha i}}^0 + I^* J_{3_{\alpha i}}^1) M_{i,1}$$
$$- (s^* J_{4_{\alpha i}}^0 + I^* J_{4_{\alpha i}}^1) M_{i,2} - (s^* J_{5_{\alpha\beta}}^0 + I^* J_{5_{\alpha\beta}}^1) Q_\beta \tag{38}$$

$$\bar{W} = W + (J_{6_i}^0 N_i + J_{7_i}^0 M_i)/m \tag{39}$$

where

$$(J_{\ell_{\alpha i}}^0, J_{\ell_{\alpha i}}^1) = \Sigma\int_k (1,z)\rho^k u_{\ell_{\alpha i}} \mathrm{d}z \quad \ell = 1,2,3,4$$

$$(J_{5_{\alpha\beta}}^0, J_{5_{\alpha\beta}}^1) = \Sigma\int_k (1,z)\rho^k u_{5_{\alpha\beta}} \mathrm{d}z$$

$$J_{6_i}^0 = \Sigma\int_k \rho^k \eta_i^k \mathrm{d}z$$

$$J_{7_i}^0 = \Sigma\int_k \rho^k \mu_i^k \mathrm{d}z \tag{40}$$

Equations (37-40) may be used to express the force and moment resultants in terms of $\bar{U}_1$, $\bar{U}_2$, $\bar{W}$, $\bar{\phi}_1$, and $\bar{\phi}_2$ and the overall equations of motion in terms of U_1, U_2, and W or vice versa. The shear equation in terms of averaged variables emerges when the derivatives of Eq. (39) with respect to x_α are added to Eq. (38).

Summary

The governing equations for the laminated plate can now be summarized. They encompass four main categories. Overall equations consist of the equations of motion (3-5) and the constitutive equations (33) and (34). In addition, two sets of equations provide the distributions of stresses and displacements in each ply. The set for stresses consists of Eqs. (20-22) and (36). The set for displacements is provided by Eqs. (24) and (29).

The overall equations of motion and constitutive equations represent a set of 11 coupled partial differential equations in terms of 3 displacement components of the plate middle surface, 3 force resultants, 3 moment resultants, and 2 shear resultants. It has been shown in Ref. 23 for homogeneous plates that, by suitably averaging the response parameters through the thickness, the Reissner's plate theory[24] equations can be obtained from stress and displacement distributions of the present type. It appears that on the basis of this relationship, one may obtain the most general sets of boundary conditions that are compatible with the system of partial differential equations from the following groups:

At $x_1 = $ const, specify either (N_1 or $\bar{U}_1$), (N_6 or $\bar{U}_2$)

(M_1 or $\bar{\phi}_1$), (M_6 or $\bar{\phi}_2$), and (Q_1 or $\bar{W}$)

At $x_2 = $ const, specify either (N_6 or $\bar{U}_1$), (N_2 or $\bar{U}_2$)

(M_6 or $\bar{\phi}_1$), (M_6 or $\bar{\phi}_2$), and (Q_2 or $\bar{W}$) (41)

The choice of the variables from each group and their combination should describe the physical nature of the restraint.

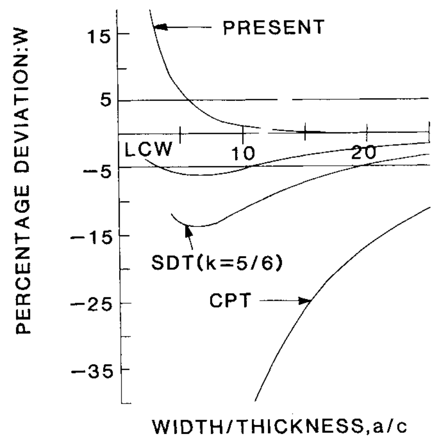

Fig. 2 Percentage deviation from the exact solution in the value of the transverse deflection at the center of a simply supported sinusoidally loaded (0,90,0) laminate ($a/b = 1$).

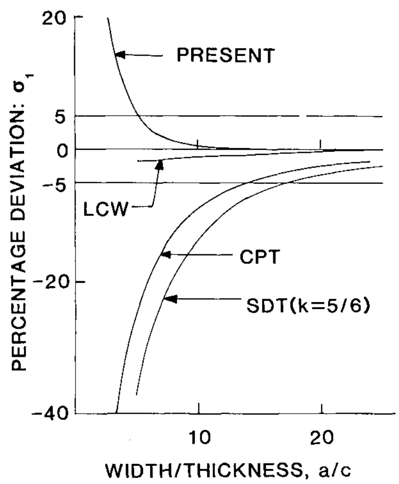

Fig. 3 Percentage deviation from the exact solution in the value of the face-parallel stress in x_1 direction at $x_1 = 0$, $x_2 = 0$, and $z = c$ for a simply supported sinusoidally loaded (0,90,0) laminate ($a/b = 1$).

In the place of averaged kinematic variables, reference-surface-related kinematic variables can also be used in specifying kinematic boundary conditions. In contrast to the more conventional displacement formulations, the situation because of the three-dimensional nature of the displacment distributions is more complex here. Experience with the use of equations and specific study of the sensitivity of predictions to boundary restraint modeling are required. The corresponding problem for homogeneous structures is discussed in Ref. 18.

Applications

The following section provides examples that illustrate the effectiveness of the new theory in the stress analysis of composite laminates. For this purpose, simply supported cross-ply (0,90,0) and quasi-isotropic (0,90,+45,−45) laminates are selected. The coordinate system is selected, for the display of results, in such a way that $-a \le x_1 \le a$, $-b \le x_2 \le b$, $-c \le z \le c$ becomes the domain of the laminate. All plies have the same thickness and the aspect ratio a/b is 1. The following material properties are assumed for each individual ply:

$$E_1 = 25 \times 10^6 \text{ psi}, \; E_2 = E_3 = 1 \times 10^6 \text{ psi},$$

$$G_{23} = 0.2 \times 10^6 \text{ psi}, \; G_{12} = G_{13} = 0.5 \times 10^6 \text{ psi},$$

$$\nu_{12} = \nu_{13} = \nu_{23} = 0.25$$

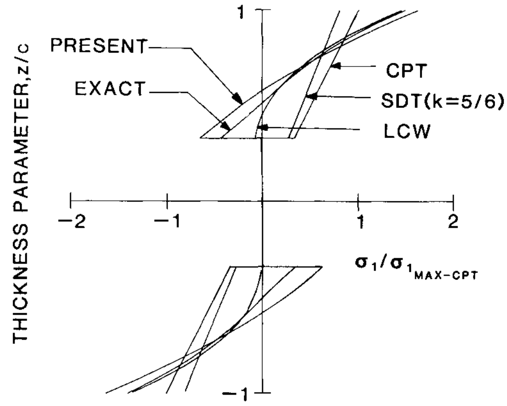

Fig. 4 Face-parallel stress in x_1 direction at $x_1 = 0$ and $x_2 = 0$ for a simply supported sinusoidally loaded (0,90,0) laminate ($a/b = 1$, $a/c = 4$).

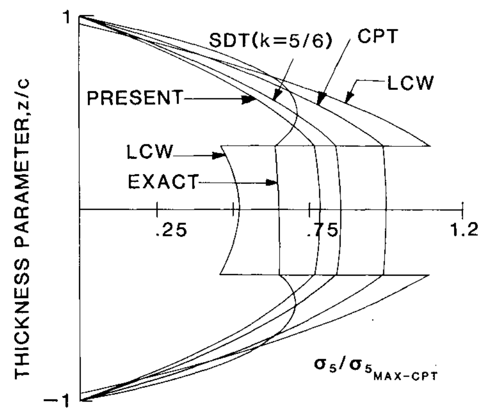

Fig. 5 Transverse shear stress in x_1-z plane at $x_1 = a$ and $n_2 = 0$, for a simply supported sinusoidally loaded (0,90,0) laminate ($a/b = 1$, $a/c = 4$).

where E is Young's modulus, G the shear modulus, and ν Poisson's ratio. These properties are acceptable for the purpose of comparison with earlier numerical results. In reality, ν_{23} should be different from both ν_{12} and ν_{13}. Both the laminates are subjected to a sinusoidally distributed pressure $q_0 \cos\pi x_1/2a \cos\pi x_2/2b$ on the upper surface $z = c$. The laminates are also considered with simple supports.

An exact solution[25] is available for the cross-ply laminate. It satisfies the following boundary conditions:

$$\text{At } x_1 = \pm a\text{:} \quad \sigma_1 = 0, \quad u_2 = 0, \quad w = 0$$

$$\text{At } x_2 = \pm b\text{:} \quad \sigma_2 = 0, \quad u_1 = 0, \quad w = 0 \qquad (42)$$

For application of the present theory, the closest approximation is to require

$$\text{At } x_1 = \pm a\text{:} \quad N_1 = 0, \quad M_1 = 0, \quad U_2 = 0, \quad \phi_2 = 0, \quad W = 0$$

$$\text{At } x_2 = \pm b\text{:} \quad N_2 = 0, \quad M_2 = 0, \quad U_1 = 0, \quad \phi_1 = 0, \quad W = 0 \qquad (43)$$

where ϕ_1 and ϕ_2 are the rotations of a normal to the laminate middle surface. The quasi-isotropic laminate is a balanced symmetric laminate. Therefore, it is assumed that the coupling stiffnesses A_{16}, A_{26}, D_{16}, and D_{26} can be neglected in solving the overall equations for the quasi-isotropic laminate. This leads to a simpler solution.

The results for the cross-ply laminate and quasi-isotropic laminate are presented in Figs. 2-7. The cross-ply laminate results are studied in a format that yields the range of validity of the theory on a function of the plate geometry. Here, the

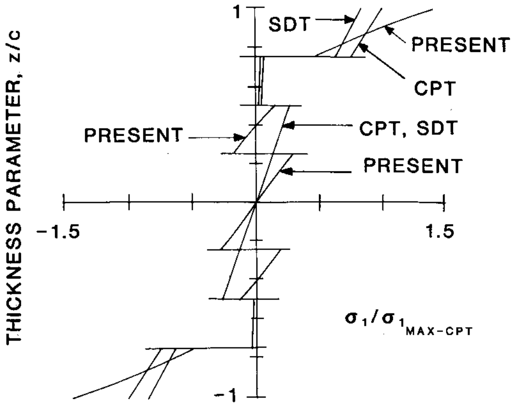

Fig. 6 Face-parallel stress in x_1 direction at $x_1 = 0$ and $x_2 = 0$ for a simply supported sinusoidally loaded $(0,90,\pm 45)_s$ laminate ($a/b = 1$, $a/c = 4$).

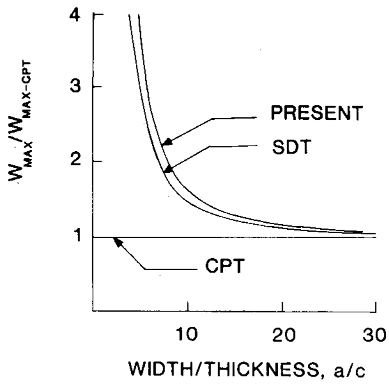

Fig. 7 Transverse deflection at the center of a simply supported sinusoidally loaded $(0,90,\pm 45)_s$ laminate ($a/b = 1$).

percentage deviations from the exact solution in the predictions for the maximum values of the response parameters are studied. The performance of the present theory is also assessed in relation to a shear deformation theory (Ref. 6, shear correction factors, 5/6) and Lo, Christensen, and Wu's higher-order theory.[14] The results from these theories are generated independently as the solutions for all response variables are not available in the respective references.

Percentage deviation from the exact solution is displayed in Fig. 2 as a function of plate geometry for the maximum transverse deflection of the cross-ply laminate. CPT and LCW denote results from classical plate theory and Lo, Christensen, and Wu's theory, respectively. The figure illustrates the predictive capability of the present theory for very thick plates. The LCW result is also in good agreement. However, it must be remembered that it is a higher-order theory and results are more difficult to obtain. In Fig. 3, predictions are compared for the maximum value of an in-plane axial stress σ_1.

The distributions of axial stress σ_1 and transverse shear stress σ_{1z} through the laminate thickness are shown in Figs. 4 and 5, respectively. A span-to-thickness ratio of 4 is used in computing the results. The agreement between the present theory and the exact solution is excellent. Of particular importance is the fact that the CPT predictions are extremely poor. The LCW shear stress distribution is also poor. The stress is discontinuous at the interfaces. This is because the stresses are computed from purely kinematic assumptions. Better results may be obtained if these stresses are computed from equilibrium equations as suggested by Lo, Christensen, and Wu in Ref. 16. However, in view of the nature of this higher-order theory, it will be difficult to compute the shear stresses in this manner. An additional reason for not using the equilibrium equations in computing the LCW transverse shear stresses is to show the type of stresses an assumed displacement formulation yields. Increased sophistication in the initial displacement assumptions alone does not lead to improved interlaminar stresses. It should be noted once again that the nonclassical surface-parallel stresses are responsible for the performance of the present theory. These stresses are not available in shear deformation theories based on linear displacement assumptions.

Results for the quasi-isotropic laminate are presented as a second example. Axial stress distribution through the thickness is shown in Fig. 6. No exact solution is available for this problem. Therefore, the shear deformation theory (SDT) is used to compare the results. It is particularly disconcerting to note SDT underestimates the axial stress in comparison to classical plate theory.

The maximum transverse deflection W_{max} is compared in Fig. 7. This clearly demonstrates the inadequacy of classical plate theory, which underestimates the deflection by as much as 25% for even significantly thin plates. The SDT result is also an underestimate.

Conclusions

A comprehensive theory for composite laminates has been presented and applied. The theory consistently accounts for nonclassical influences related to transverse shear strains, transverse normal strain, and nonclassical surface-parallel stresses. A benchmark solution illustrates its exceptional predictive capability in laminate stress analysis.

Appendix

Let z_k denote the coordinate of the upper surface of the kth ply as shown in Fig. 1, δ_{ij} denote a Kronecker delta,

$$\delta_{ij} = 1 \text{ if } i=j; \quad i,j = 1,2,6$$
$$= 0 \text{ if } i \neq j; \quad i,j = 1,2,6 \tag{A1}$$

and

$$m^*/I = -s^*/s = I^*/m = 1/(mI - s^2) \tag{A2}$$

The functions of the thickness coordinate in Eqs. (20-22) may be written as

$$\bar{n}_{ij}^k = \bar{n}_{ij}^k(z_k) + (\bar{A}_{ij}^k - \rho^k m^* \delta_{ij})(z - z_k)$$
$$+ \tfrac{1}{2}(\bar{B}_{ij}^k - \rho^k s^* \delta_{ij})(z^2 - z_k^2)$$
$$\bar{m}_{ij}^k = \bar{m}_{ij}^k(z_k) + (\bar{B}_{ij}^k - \rho^k s^* \delta_{ij})(z - z_k)$$
$$+ \tfrac{1}{2}(\bar{D}_{ij}^k - \rho^k I^* \delta_{ij})(z^2 - z_k^2)$$
$$\bar{t}^k = \bar{t}^k(z_k) + \rho^k s^*(z - z_k) + \tfrac{1}{2}\rho^k I^*(z^2 - z_k^2)$$
$$\tilde{n}_{ij}^k = \tilde{n}_{ij}^k(z_k) + \bar{n}_{ij}^k(z_k)(z - z_k)$$
$$+ (\bar{A}_{ij}^k - \rho^k m^* \delta_{ij})\tfrac{1}{2}(z^2 - 2zz_k + z_k^2)$$
$$+ (\bar{B}_{ij}^k - \rho^k s^* \delta_{ij})(1/6)(z^3 - 3zz_k^2 + 2z_k^3)$$
$$\tilde{m}_{ij}^k = \tilde{m}_{ij}^k(z_k) + \bar{m}_{ij}^k(z_k)(z - z_k)$$
$$+ (\bar{B}_{ij}^k - \rho^k s^* \delta_{ij})\tfrac{1}{2}(z^2 - 2zz_k + z_k^2)$$
$$+ (\bar{D}_{ij}^k - \rho^k I^* \delta_{ij})(1/6)(z^3 - 3zz_k^2 + 2z_k^3)$$
$$\tilde{t}^k = \tilde{t}^k(z_k) + \bar{t}^k(z_k)(z - z_k) + \rho^k s^* \tfrac{1}{2}(z^2 - 2zz_k + z_k^2)$$
$$+ \rho^k I^*(1/6)(z^3 - 3zz_k^2 + 2z_k^3) + (\rho^k/m^*)(z - z_k)$$
$$\tilde{t}_I^k = \tilde{t}_I^k(z_k) + (\rho^k/m^*)(z - z_k) \tag{A3}$$

The values of the above functions at z_k may be obtained from the continuity requirements; for example,

$$\bar{n}_{ij}^k(z_k) = \sum_{p=1}^{k-1} [(\bar{A}_{ij}^p - \rho^p m^* \delta_{ij}(z_{p+1} - z_p) + \tfrac{1}{2}(\bar{B}_{ij}^p - \rho^p s^* \delta_{ij}(z_{p+1}^2 - z_p^2))] \tag{A4}$$

Let ϕ denote the particular ply which contains $z=0$ surface. Then

$$\eta_i^k(z) = \sum_{p=1}^{k-1} \int_{z_p}^{z_{p+1}} S_{3j}^p n_{ij}^p(\zeta)\mathrm{d}\zeta - \sum_{p=1}^{\phi-1} \int_{z_p}^{z_{p+1}} S_{3j}^p n_{ij}^p(\zeta)\mathrm{d}\zeta - \int_{z_\phi}^{0} S_{3j}^\phi n_{ij}^\phi(\zeta)\mathrm{d}\zeta + \int_{z_k}^{z} S_{3j}^k n_{ij}^k(\zeta)\mathrm{d}\zeta \tag{A5}$$

The remaining functions of the thickness coordinate appearing in displacement expressions can be obtained by appropriately replacing the arguments in the above integrals.

Acknowledgment

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